years and was chosen a delegate to the General Assembly, meeting at Portland, Oregon, in May, 1892, serving the church in this capacity with fidelity and intelligence. In this biography of Professor Hoover, there is a valuable lesson to be learned. It is this: energy and perseverence will bring a sure reward to earnest effort. We see how the clerk in a county-seat store, in embarassing circumstances and unknown to the world of thinkers, became the well known Professor of Mathematics and Astronomy in one of the leading Institutions of learning in the State of Ohio. "Not to know him argues yourself unknown."

[From *Finkel's Mathematical Solution Book.*]

APPLICATION OF THE NEW EDUCATION TO THE DIFFERENTIAL AND INTEGRAL CALCULUS.

By **FLETCHER DURELL, Ph. D., Professor of Mathematics, Dickinson College, Carlisle, Pennsylvania.**

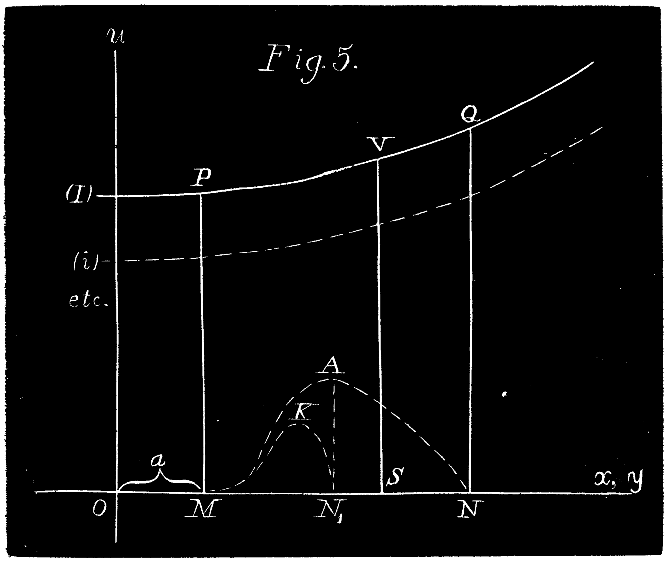
[Continued from the January Number]

If the quantity which has been represented by curves may also be regarded as existing independently of any spacial arrangement, its magnitude and magnitude relations in both cases being the same, the formulas of differentiation obtained above apply to both; that is, they apply to functions as well as to curves. The student may at once be brought to realize the greater flexibility and freedom of treatment obtained by using them functionally. We thus arrive at the more general definition of differentiation, $\dfrac{dy}{dx}=\text{limit}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$.

The process of determining the slopes of curves by the above geometrical method, and the use of the related variables as auxiliary quantities determining the slopes by exact contact, and the practice in constructing tangents to the curves by the use of these slopes, build up firm and exact and vivid conceptions of the quantities dealt with. When the student comes to take up the more general idea of functional quantities, arranged irregularly or indefinitely in space, the geometrical conceptions already formed aid in giving firmness and reality to the quantities dealt with as differential coefficient, and a sense of the absolute precision of their values as determined by variables moving up into contact with them.

However at the outset of each division of the subject, as in dealing with partial differentiation, series, indeterminate expressions so-called, etc, it is best to establish properties in the geometrical form if only for the double light that is thrown on them. Space will not permit us to show in detail how this is done, and we will but illustrate these further applications of the method by giving a proof of Taylor's Formula with Remainder. In Fig. 5, let (I), or PQ, be the section of the surface, $u=f(x+y)$ made by the ux-plane. Since this surface slopes in the same way from the xy—plane, as it does from the uy—plane, this one trace may be taken as an adequate representation of the whole surface for the present purpose.

In the equation $u=f(x+y)$, let x have some temporarily constant value as a, (OM). Let $u=f(a+y)$ be such a function that when y varies between zero and $h(MN)$ in value, $u=f(a+y)$ and its first $n+1$ derived* lines are all continuous. Then QN, that is, $f(a+h)$ may be expressed thus,† $f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\ldots+\frac{h^n}{\lfloor n}f^n(a)+\frac{h^{n+1}}{\lfloor n+1}R\ldots$



(4). R, in this equation, is an unknown quantity (constant as long as a and h are) and must be determined so as to satisfy equation (4). This we now proceed to do.

If we use another unknown quantity v and denote MS by y, we may express VS thus, $f(a+y)=f(a)+yf'(a)+\frac{y^2}{\lfloor 2}f''(a)+\ldots+\frac{y^n}{\lfloor n}f^n(a)+\frac{y^{n+1}}{\lfloor n+1}R+v\ldots(5)$. Taking y as a variable and v as function, this equation traced will give a curve MAN, which must pass through M and N, since in (4), when $y=o$, $v=o$, and also when $y=h$, $v=o$, by aid of (4). Therefore‡ if we denote the slope of MAN by $\frac{dv}{dy}$ or v', $v'=o$ for some value of y between zero and h, as MN_1.

Differentiate (5) with respect to y, $f'(a+y)=f'(a)+yf''(a)+\ldots+\frac{y^{n-1}}{\lfloor n-1}f^n(a)+\frac{y^n}{\lfloor n}R+v'\ldots(6)$. In (6), when $y=o$, $v'=o$. $\therefore$ (6) may be traced as a curve MKN_1 passing through M and N_1 and somewhere as at K having its slope $v''=o$. Differentiate (6) and proceed as before; after $n+1$ processes we obtain $f^{n+1}(a+y)=R+\frac{d^{n+1}v}{dy^{n+1}}\ldots(7)$, in which $\frac{d^{n+1}v}{dy^{n+1}}$, being the slope of the preceding line, must $=o$, for some value of y, as $y=\theta h$, where θ is a positive proper fraction; $\therefore$ in (4), let $y=\theta h$, then since $\frac{d^{n+1}v}{dy^{n+1}}=o$, $f^{n+1}(a+\theta h)=R\ldots(8)$.

* By a derived line is meant one such that each ordinate of it equals the slope of the preceding line at the corresponding point. On the figure, (i) is drawn as the 1st derived of (I).

† See Todhunter's Differential Calculus.

‡ Using the principle that if a curve cross the axis of x at two points, and if the curve and its first derived line be continuous between these points, the slope of the curve at some point between the two points of crossing, must be zero.

That is, $R=$ slope of the nth derived line of $u=f(a+y)$ at that point whose abscissa is $a+\theta h$. Substituting in (4) for R we obtain

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\underline{|2}}f''(a)+\ldots.+\frac{h^n}{\underline{|n}}f^n(a)+\frac{h^{n+1}}{\underline{|n+1}}f^{n+1}(a+\theta h).$$

What is true for one set of values for a and h is true for others, provided the above conditions of continuity etc. are satisfied. Hence, we may write the general formula,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\underline{|2}}f''(x)+\ldots.+\frac{h^n}{\underline{|n}}f^n(x)+\frac{h^{n+1}}{\underline{|n+1}}f^{n+1}(x+\theta h).$$

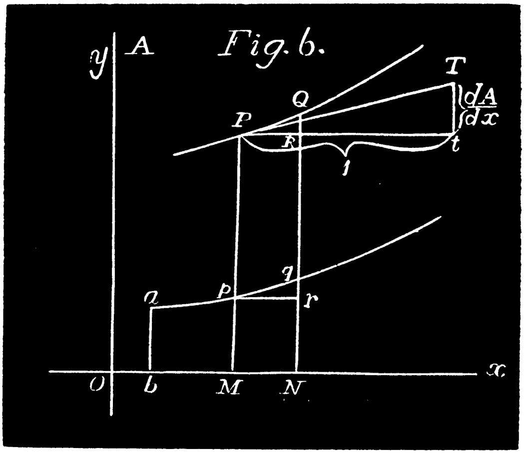
After the student realizes the advantages which come from the use of the functional idea, he is to be introduced to the further advantages which come from the use of infinitesimals. This method is to be taught as an abbreviation of the preceding methods. In the first place, we observe that certain terms which contain higher powers of Δx and Δy (or other increments) will disappear when the secant becomes a tangent (or the limit is taken), hence, we cancel them out by anticipation. The infinitesimal is an aid in determining these. Often we do not actually develope quantities into these higher powers of the increments, but determine by the use of the infinitesimal idea, that they would be such if developed. Again, if we neglect infinitesimals of higher orders and all the terms that would be produced by them in the course of the work, the neglected terms being kept in mind along with the thought that they will disappear whenever we desire to make the tangent reduction or pass to the limit, and if we retain Δy and Δx, as dy and dx and operate with them, still other advantages result. All these various advantages may be grouped in three classes:

First, in expressing results. Several equations are often expressed as one. Thus $\frac{dl}{dx}=\sqrt{1+\left\{\frac{dy}{dx}\right\}^2}$ and $\frac{dl}{dy}=\sqrt{1+\left\{\frac{dx}{dy}\right\}^2}$ are expressed in the single equation $dl=\sqrt{dx^2+dy^2}$. Likewise, $dl=\sqrt{dx^2+dy^2+dz^2}$ stands for three equations.

Second, in proving properties, as in cases where infinitesimals of higher orders are neglected at the outset.

Third, in suggesting properties.

In all the first cases in which infinitesimals are used, the proof should be given in the unabreviated geometric form first. Then by way of comparison, the abbreviated form by the use of infinitesimals and the resulting advantages are to be presented. This may be illsutrat-



ed by proving that the differential coefficient of the area of a curve $y=f(x)$, is $\frac{dA}{dx}=y$. In Fig. 6, let the equation of apq be $y=f(x)$ and let its areas be measured from a given ordinate ab. Let PQ be such a curve that the number of linear spaces in any ordinate = the number of units of area from ab up to the corresponding ordinate. Thus, the number of linear units in PM=number of units of area in $abpM$, and number of linear units in QM=number of units of area in $abqN$.

If we denote the ordinates of PQ by A, $QR=\Delta A$=the number of linear units in $pqMN$, $\therefore \Delta A=y\Delta x+\theta\Delta x\Delta y$, where θ is a proper fraction.

$\therefore \frac{\Delta A}{\Delta x}=y+\theta\Delta x$, and $\frac{dA}{dx}=y$, that is $Tt=pM$, by the usual reduction. But this process is abbreviated by the use of infinitesimals thus: if MN is diminished till it is an infinitesimal, dx, $pqMN$ is an infinitesimal of area, dA. $\therefore dA=ydx+\theta dxdy=ydx$, neglecting the infinitesimal of the 2nd order.

As a result of this method of taking up the Differential Calculus, the process of integration has two aspects; first, the simple concrete aspect of unsloping; second, the aspect of summation often convenient in practice. If the slope of a curve at every point=twice the abscissa of the point, that is $\frac{dy}{dx}=2x$, it is a matter of simple recollection that the equation of the curve is $y=x^2+c$. From previous constructions in the Differential Calculus, the two curves rise together as a picture with the slopes of one identical in magnitude with the ordinates of the other. The relation can be varified anew by constructions.

This sharp vividness in the unsloping aspect aids in grasping integration as summation. Thus if the summation be that of the areal elements of a curve, their sum is represented by a straight line, which approaches $\frac{dA}{dx}$ as h (or Δx) approaches zero, hence we get here the same *contact* of the variable with the limit which it determines, as we did in the Differential Calculus.

The conception is also useful in grasping multiple integration as a summation. Thus to integrate $\int_c^\gamma\int_b^\beta\int_a^\alpha f(x,y,z)\,dxdydz$, dx, dy, dz being infinitesimals, we conceive the space between the limits of the integration, to be divided by parallel planes into parallelopipeds, whose edges are dx, dy, and dz. Denote the value of $f(x, y, z)$ by which each small parallelopiped element, $dxdydz$, is multiplied as the *radiant* of that parallelopiped. Conceive the radiant of each parallelopiped to extend from each and all points of it as a bundle or sheaf of equal and parallel straight lines (it may perhaps be helpful to think of the radiants as lines of force). Hence, the aggregate, $\Sigma f(x,y,z)\,dxdydz$, denotes a set of parallel bundles of radiants, the individual members of each bundle being equal among themselves. Hence, this aggregate is discontinuous in three directions. Then it may readily be shown that a single integration (unsloping) makes this aggregate continuous in one direction, and the three successive integrations make it continuous in all directions; the result being an aggregate in which each

point has a radiant determined by the co-ordinates of the point, and the entire aggregate being equal to the volume determined by the limits of the integration multiplied by the average size of the radiants.

This method of presenting the first principles of the Differential and Integral Calculus is in accord with, indeed is developed from the principles of the new education. The method proceeds from the known to the unknown by continuous steps. The idea of slope is already firmly established in the mind of the student from the study of Analytical Geometry. This idèa is developed into those of the differential coefficient and the infinitesimal without any sudden leap to functions and abstract continuous quantity. The method proceeds from the concrete to the abstract. It gives the student at the outset something which he can see, make, and count and hence, developes his self activity. He proceeds to other ideas not mechanically or under dictation but for the sake of clear realized advantages. The advantage of geometrical conceptions at the foundation of a subject and their pervasive, illuminating power is well expressed by Prof. Simon Newcomb thus:* "A practice has come into vogue among professional mathematicians which illustrates the difficulty I have mentioned, and the way in which it may be avoided. The relations among imaginary quantities in algebra are so much more complex than those among real quantities that they evade the direct comprehension of even the most expert mathematician. The result was that no progress was made in the study of a subject now at the basis of a large part of mathematics, until Gauss and Cauchy conceived the happy idea of representing the two elements which enter into an imaginary algebraic quantity by the position of a point on a plane. The motion of the point embodied the idea of the variation of the quantity, and the study of the subject was thus reduced to the study of the motion of points; an abstraction was replaced by a concrete representation. The result was that, in the conception of writers on the subject, position speedily took the place of quantity; the word "point" replaced the word "value", and in this way an extended branch of mathematics was constructed, which would not have been possible had the abstract variable of algebra not been replaced by the moving point of geometry. If the greatest mathematical minds feel such an aid to be necessary to their work, why should not a corresponding aid be offered to the farmer's boy, who is engaged, year after year, in struggling with numbers, of whose relations to sensible objects he can have no clear conception", and, we may add, to the student who with no extended preliminary mathematical training takes up the study of the Differential and Integral Calculus, the most difficult perhaps in the range of the curriculum? Furthermore, as the first conceptions of the theory of functions are now usually given in the geometric form, one of the advantages of the method of teaching the Calculus here presented is that it thus also opens the way to the general treatment of the subject where the complex variable is included.

*See Educational Review, Vol. IV, p. 279.